Journal of Sound and Vibration (2000) **235**(1), 153–155
doi:10.1006/jsvi.1999.2829, available online at http://www.idealibrary.com on IDEAL®

LETTERS TO THE EDITOR

THE ADVANTAGE OF AN IMPLICIT CORRECTOR AT BIFURCATION

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(*Received* 27 *December* 1999)

1. INTRODUCTION

The debate over the use of explicit versus implicit finite difference schemes for the numerical solution [1] of initial value problems is principally about the balance between the conflicting predicaments of stability and computational efficiency. Explicitly, only conditionally stable, predictors do not require repeated, costly, and occasionally uncertain, numerical solution of a non-linear equation for stepping out in time, but place severe restrictions on the size of the time step as a precaution against explosive instabilities.

Yet differential equations that initially bifurcate into several branches sharing a common incipient tangent are only partly solved by explicit methods that have no means of discriminating between the different emanating offshoots of the non-unique solution. In case the differential equation of motion can be factored, then each factor may be solved separately, but if applied naively, the explicit scheme will follow only one branch of the solution, that may even be of secondary interest, ignoring all others.

The implicit corrector, because it samples the solution away from the bifurcation, is apt to place points on all branches that grow out of a singular starting point.

2. PRODUCT EQUATIONS

Consider the pair of initial value problems

$$x'' + x + \alpha x^3 = 0 \quad \text{and} \quad x' = 0, \quad x(0) = 1, \quad x'(0) = 0 \tag{1}$$

in which $x = x(t)$, and where $(\ )'$ means differentiation with respect to time t. System (1) includes two equations of motion that share the same initial conditions of position and velocity. The product, non-linear, initial value problem

$$x'(x'' + x + \alpha x^3) = 0, \qquad x(0) = 1, \qquad x'(0) = 0 \tag{2}$$

has two solutions bifurcating at $t = 0$, one periodic and one of rest.

The first equation in system (1) describes the motion of a mass–spring system with a non-linear conservative restoring force and we readily obtain for the combined equation (2) the first integral

$$x'^2 + x^2 + \tfrac{1}{2}\alpha x^4 = 1 + \tfrac{1}{2}\alpha, \qquad x(0) = 1, \qquad x'(0) = 0 \tag{3}$$

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expressing conservation of mechanical energy in the vibrating system. Equation (3) may be rewritten as

$$x' = -\sqrt{1 - x^2 + \tfrac{1}{2}\alpha(1 - x^4)} \quad \text{or} \quad x' = f(x). \tag{4}$$

The non-uniqueness of the solution to the initial value problem is evident from the acceleration

$$x'' = \frac{\partial f}{\partial x}x' = \frac{x + \alpha x^3}{\sqrt{(1 - x^2) + \tfrac{1}{2}\alpha(1 - x^4)}}x' \tag{5}$$

that becomes ambiguous at $t = 0$.

Another product equation of interest, this time with a non-algebraic non-linearity, is that of the mathematical pendulum and constant velocity that combine into

$$x'(x'' + \omega^2 \sin x) = 0, \quad \omega > 0, \quad x(0) = \alpha, \qquad x'(0) = 0. \tag{6}$$

Integration with respect to time readily leads to the energy equation

$$\tfrac{1}{2}x'^2 - \omega^2 \cos x = -\omega^2 \cos \alpha, \qquad x(0) = \alpha, \qquad x'(0) = 0, \tag{7}$$

which is first order, non-linear and singular at $x = 0$.

3. NUMERICAL SOLUTION

Attempting to numerically solve the energy balance equation (3) by the explicit Euler method $x_1 = x_0 + \tau x_0'$, $x = x_0$, where $x_1 = x(\tau)$, approximately, and $x_0 = x(0)$, we get $x_1 = 1$ and consequently $x_n = x(n\tau) = 1$ which is only the rest solution of equation (2). The explicit method misses the periodic solution.

Use of the higher order Euler predictor $x_1 = x_0 + \tau x_0' + \tfrac{1}{2}\tau^2 x_0''$ is impossible here because of the ambiguity of $x_0'' = x''(0)$.

The explicit Euler scheme $x' = (x_1 - x_0)/\tau$, $x = x_1$, recasts equation (3) into the algebraic form

$$(x_1 - 1)^2/\tau^2 + x_1^2 + \tfrac{1}{2}\alpha x_1^4 = 1 + \tfrac{1}{2}\alpha \tag{8}$$

or

$$(x_1 - 1)^2 + \tau^2(x_1^2 - 1) + \tfrac{1}{2}\tau^2\alpha(x_1^4 - 1) = 0 \tag{9}$$

that is seen to contain the factor $x_1 - 1$ and hence the solution $x(t) = 1$ for the state of rest. Factoring out $x_1 - 1$ from equation (1) leaves us with

$$x_1(1 + \tau^2) - 1 + \tau^2 + \tfrac{1}{2}\alpha\tau^2(x_1 + 1)(x_1^2 + 1) = 0 \tag{10}$$

which is third order in x_1.

In case $\alpha = 0$, equation (10) is readily solved in closed form to produce

$$x_1 = \frac{1 - \tau^2}{1 + \tau^2} = 1 - 2\tau^2 + \cdots \tag{11}$$

as compared with the exact

$$x_1 = x(\tau) = \cos\tau = 1 - \tfrac{1}{2}\tau^2 + \cdots \tag{12}$$

indicating that the one-step error in the approximate Euler solution is $O(\tau^2)$.

Once the periodic branch is discovered by the implicit corrector, solution of the initial value problem may proceed by explicit means. Only one leap of the implicit method is needed to step out of the bifurcation point on any branch that grows out of it, mitigating the difficulties associated with the use of such schemes.

One may be tempted to use the higher order corrector $x' = (x_1 - x_0)/\tau$, $x = (x_1 + x_0)/2$ in equation (3) for a possibly greater accuracy for $x_1 = x_1(\tau)$. Doing that we get, assuming $\alpha = 0$ for the sake of expository simplicity, the algebraic equation

$$(x_1 - 1)^2 + \tfrac{1}{4}\tau^2(x_1 + 1)^2 = \tau^2 \tag{13}$$

that still includes the factor $x_1 - 1$, or the solution $x(t) = 1$. Factoring out x_1 reduces equation (13) to the linear

$$x_1(1 + \tfrac{1}{4}\tau^2) - 1 + \tfrac{3}{4}\tau^2 = 0 \tag{14}$$

and another solution emerges from it,

$$x_1 = \frac{1 - \frac{3}{4}\tau^2}{1 + \frac{1}{4}\tau^2} = 1 - \tau^2 + \cdots, \tag{15}$$

with an approximation error that is still only $O(\tau^2)$ because of the singularity at $t = 0$.

REFERENCES

1. FRIED 1979 *Numerical Solutions of Differential Equations*. New York, NY: Academic Press.